Bastard silly integral

Maple evaluates our definite integral to be $\frac{\pi\sqrt{6}}{6}$, and takes 6 seconds exactly doing this calculation.

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> f:=1/(7-5*sin(x));
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$$f := \frac{1}{7 - 5\sin(x)}$$

```
> Int(f,x=0..2*Pi);
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$$\int_0^{2\pi} \frac{1}{7 - 5\sin(x)}\,dx$$

```
> int(f,x=0..2*Pi);
```

$$\frac{\pi\sqrt{6}}{6}$$

Mattywoo's Solution

$$\int_0^{2\pi} \frac{1}{7 - 5\sin(x)}\,\mathrm{d}x$$

Let $u = \sin x$

$$\begin{aligned} u &= \sin x \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= \cos x \\ &= \sqrt{1-u^2} \end{aligned}$$

Our indefinite integral becomes

$$\int \frac{1}{(7-5u)\sqrt{1-u^2}}\,\mathrm{d}u$$

We can integrate this by splitting the expression into partial fractions, since the the integral of a sum is equivilant to the sum of each integrated term.

$$\begin{aligned} \frac{1}{(7-5u)\sqrt{1-u^2}} &= \frac{A}{(7-5u)} + \frac{Bu+C}{\sqrt{1-u^2}} \\ 1 &= A\sqrt{1-u^2} + (Bu+C)(7-5u) \end{aligned}$$

Let $u = \frac{7}{5}$

$$\begin{aligned}
1 &= A\sqrt{1-\frac{49}{25}} + (\frac{7}{5}B+C)(0) \\
&= A\sqrt{-1}\frac{\sqrt{24}}{\sqrt{25}} \\
&= \frac{2i\sqrt{6}}{5}A \\
A &= \frac{5}{2i\sqrt{6}}
\end{aligned}$$

Rationalising the denominator yelids

$$A = \frac{-5i\sqrt{6}}{12}$$

Let $u = 0$

$$\begin{aligned}
1 &= A + 7C \\
7C &= 1 - A \\
C &= \frac{1}{7} + \frac{5i\sqrt{6}}{84} \\
C &= \frac{12+5i\sqrt{6}}{84}
\end{aligned}$$

Let $u = 1$

$$\begin{aligned}
1 &= 2(B+C) \\
2B &= 1 - 2C \\
B &= \frac{1}{2} - C \\
B &= \frac{30-5i\sqrt{6}}{84}
\end{aligned}$$

Substituting A, B and C back into our expression in partial fractions gives

$$\begin{aligned}
\frac{1}{(7-5u)\sqrt{1-u^2}} &= \frac{\left(\frac{-5i\sqrt{6}}{12}\right)}{(7-5u)} + \frac{\left(\frac{30-5i\sqrt{6}}{84}\right)u + \left(\frac{12+5i\sqrt{6}}{84}\right)}{\sqrt{1-u^2}} \\
&= \frac{-5i\sqrt{6}}{12(7-5u)} + \frac{(30-5i\sqrt{6})u + (12+5i\sqrt{6})}{84\sqrt{1-u^2}} \\
&= \frac{-5i\sqrt{6}}{84-60u)} + \frac{(30-5i\sqrt{6})u}{84\sqrt{1-u^2}} + \frac{(12+5i\sqrt{6})}{84\sqrt{1-u^2}}
\end{aligned}$$

So our integral now looks like

$$\int \frac{1}{(7-5u)\sqrt{1-u^2}}\,\mathrm{d}u = \left(\frac{i\sqrt{6}}{12}\right)\ln(84-60u) + \int \frac{(30-5i\sqrt{6})u}{84\sqrt{1-u^2}}\,\mathrm{d}u + \frac{12+5i\sqrt{6}}{84}\arcsin u + k$$

where k is an abritrary constant.

Re-substitute $u = \sin x$ and $\mathrm{d}u = \cos x\,\mathrm{d}x$

$$\begin{aligned}\int \frac{1}{7-5\sin(x)}\,\mathrm{d}x &= \left(\frac{i\sqrt{6}}{12}\right)\ln(84-60\sin x) + \int \frac{(30-5i\sqrt{6})\sin x\cos x}{84\cos x}\,\mathrm{d}x + \frac{12+5i\sqrt{6}}{84}x + k \\ &= \left(\frac{i\sqrt{6}}{12}\right)\ln(84-60\sin x) + \frac{(5i\sqrt{6}-30)\cos x}{84} + \frac{12+5i\sqrt{6}}{84}x + \gamma\end{aligned}$$

where γ is an arbitrary constant including k.

$$\begin{aligned}\int_0^{2\pi} \frac{1}{7-5\sin(x)}\,\mathrm{d}x &= \left[\left(\frac{i\sqrt{6}}{12}\right)\ln 84 + \frac{(5i\sqrt{6}-30)}{84} + \frac{12+5i\sqrt{6}}{84}2\pi + \gamma\right] \\ &- \left[\left(\frac{i\sqrt{6}}{12}\right)\ln 84 + \frac{(5i\sqrt{6}-30)}{84} + \gamma\right] \\ \int_0^{2\pi} \frac{1}{7-5\sin(x)}\,\mathrm{d}x &= \frac{12+5i\sqrt{6}}{42}\pi\end{aligned}$$

So the magnitude of the definite integral is the required solution

$$\begin{aligned}\left|\frac{12+5i\sqrt{6}}{42}\pi\right| &= \frac{\sqrt{144+25\times 6}\,\pi}{42} \\ &= \frac{\sqrt{294}\,\pi}{42} \\ &= \frac{\sqrt{49}\sqrt{6}\,\pi}{7\times 6} \\ &= \frac{\pi\sqrt{6}}{6}\end{aligned}$$

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> plot(f, x=0..2*Pi, 0..0.55);
```